Exercise 2.5.7

Solve Laplace's equation inside a 60° wedge of radius a subject to the boundary conditions [*Hint*: In polar coordinates,

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0,$$

it is known that if $u(r,\theta) = \phi(\theta)G(r)$, then $\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = -\frac{1}{\phi}\frac{d^2\phi}{d\theta^2}$.]:

(a) $\quad u(r,0) = 0, \qquad u\left(r,\dfrac{\pi}{3}\right) = 0, \qquad u(a,\theta) = f(\theta)$

(b) $\quad \dfrac{\partial u}{\partial \theta}(r,0) = 0, \qquad \dfrac{\partial u}{\partial \theta}\left(r,\dfrac{\pi}{3}\right) = 0, \qquad u(a,\theta) = f(\theta)$

Note: The variable a should be italicized.

Solution

Because the Laplace equation and all but one of its associated boundary conditions are linear and homogeneous, the method of separation of variables can be applied to solve it. Assume a product solution of the form $u(r,\theta) = R(r)\Theta(\theta)$ and plug it into the PDE.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}R(r)\Theta(\theta)\right] + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}R(r)\Theta(\theta) = 0$$

$$\frac{\Theta(\theta)}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{R(r)}{r^2}\frac{d^2\Theta}{d\theta^2} = 0$$

Multiply both sides by $r^2/[R(r)\Theta(\theta)]$ in order to separate variables.

$$\frac{r}{R(r)}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{1}{\Theta(\theta)}\frac{d^2\Theta}{d\theta^2} = 0$$

$$\frac{r}{R(r)}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = -\frac{1}{\Theta(\theta)}\frac{d^2\Theta}{d\theta^2}$$

The only way a function of r can be equal to a function of θ is if both are equal to a constant λ.

$$\frac{r}{R(r)}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = -\frac{1}{\Theta(\theta)}\frac{d^2\Theta}{d\theta^2} = \lambda$$

As a result of separating variables, the PDE has reduced to two ODEs—one in each independent variable.

$$\left.\begin{array}{r} \dfrac{r}{R}\dfrac{d}{dr}\left(r\dfrac{dR}{dr}\right) = \lambda \\[2mm] -\dfrac{1}{\Theta}\dfrac{d^2\Theta}{d\theta^2} = \lambda \end{array}\right\}$$

Values of λ for which nontrivial solutions to these ODEs and the associated boundary conditions exist are called eigenvalues, and the solutions themselves are called eigenfunctions. Note that it doesn't matter whether the minus sign is grouped with r or θ as long as all eigenvalues are taken into account.

Part (a)

Substitute the product solution $u(r, \theta) = R(r)\Theta(\theta)$ into the homogeneous boundary conditions.

$$u(r, 0) = 0 \quad \rightarrow \quad R(r)\Theta(0) = 0 \quad \rightarrow \quad \Theta(0) = 0$$
$$u\left(r, \frac{\pi}{3}\right) = 0 \quad \rightarrow \quad R(r)\Theta\left(\frac{\pi}{3}\right) = 0 \quad \rightarrow \quad \Theta\left(\frac{\pi}{3}\right) = 0$$

Solve the ODE for Θ.

$$\frac{d^2\Theta}{d\theta^2} = -\lambda\Theta$$

Check to see whether there are positive eigenvalues: $\lambda = \mu^2$.

$$\frac{d^2\Theta}{d\theta^2} = -\mu^2\Theta$$

The general solution can be written in terms of sine and cosine.

$$\Theta(\theta) = C_1 \cos\mu\theta + C_2 \sin\mu\theta$$

Apply the boundary conditions to determine C_1 and C_2.

$$\Theta(0) = C_1 = 0$$
$$\Theta\left(\frac{\pi}{3}\right) = C_1 \cos\mu\frac{\pi}{3} + C_2 \sin\mu\frac{\pi}{3} = 0$$

This first equation makes the second equation reduce to $C_2 \sin\mu\frac{\pi}{3} = 0$. To avoid the trivial solution, we insist that $C_2 \neq 0$.

$$\sin\mu\frac{\pi}{3} = 0$$
$$\mu\frac{\pi}{3} = n\pi, \quad n = 1, 2, \ldots$$
$$\mu = 3n$$

There are positive eigenvalues $\lambda = (3n)^2$, and the eigenfunctions associated with them are

$$\Theta(\theta) = C_2 \sin\mu\theta \quad \rightarrow \quad \Theta_n(\theta) = \sin 3n\theta.$$

Using $\lambda = 9n^2$, solve the ODE for R now.

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = 9n^2$$

Expand the left side.

$$\frac{r}{R}(R' + rR'') = 9n^2$$

Multiply both sides by R and bring all terms to the left side.

$$r^2R'' + rR' - 9n^2R = 0$$

This is an equidimensional ODE, so it has solutions of the form $R(r) = r^m$.

$$R = r^m \quad \rightarrow \quad R' = mr^{m-1} \quad \rightarrow \quad R'' = m(m-1)r^{m-2}$$

Substitute these formulas into the ODE and solve the resulting equation for m.

$$r^2 m(m-1)r^{m-2} + rmr^{m-1} - 9n^2 r^m = 0$$

$$m(m-1)r^m + mr^m - 9n^2 r^m = 0$$

$$m(m-1) + m - 9n^2 = 0$$

$$m^2 - 9n^2 = 0$$

$$(m+3n)(m-3n) = 0$$

$$m = \{-3n, 3n\}$$

Two solutions to the ODE are $R = r^{-3n}$ and $R = r^{3n}$. By the principle of superposition, the general solution for R is a linear combination of these two.

$$R(r) = Ar^{-3n} + Br^{3n}$$

Now check to see if zero is an eigenvalue.

$$\frac{d^2\Theta}{d\theta^2} = 0$$

The general solution is a straight line.

$$\Theta(\theta) = C_3\theta + C_4$$

Apply the two boundary conditions to determine C_3 and C_4.

$$\Theta(0) = C_4 = 0$$
$$\Theta\left(\frac{\pi}{3}\right) = C_3\frac{\pi}{3} + C_4 = 0$$

This first equation makes the second equation reduce to $C_3\frac{\pi}{3} = 0$, which makes $C_3 = 0$.

$$\Theta(\theta) = 0$$

The trivial solution is obtained, so zero is not an eigenvalue. Check to see if there are negative eigenvalues: $\lambda = -\gamma^2$.

$$\frac{d^2\Theta}{d\theta^2} = \gamma^2\Theta$$

The general solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$\Theta(\theta) = C_5\cosh\gamma\theta + C_6\sinh\gamma\theta$$

Apply the two boundary conditions to determine C_5 and C_6.

$$\Theta(0) = C_5 = 0$$
$$\Theta\left(\frac{\pi}{3}\right) = C_5\cosh\gamma\frac{\pi}{3} + C_6\sinh\gamma\frac{\pi}{3} = 0$$

This first equation makes the second equation reduce to $C_6\sinh\gamma\frac{\pi}{3} = 0$. No nonzero value of γ can satisfy this equation, so $C_6 = 0$.

$$\Theta(\theta) = 0$$

The trivial solution is obtained, so there are no negative eigenvalues. According to the principle of superposition, the general solution to the PDE is a linear combination of the eigenfunctions over all the eigenvalues.

$$u(r, \theta) = \sum_{n=1}^{\infty} (A_n r^{-3n} + B_n r^{3n}) \sin 3n\theta$$

For the solution to remain finite as $r \to 0$, set $A_n = 0$.

$$\boxed{u(r, \theta) = \sum_{n=1}^{\infty} B_n r^{3n} \sin 3n\theta}$$

Use the boundary condition at $r = a$ to determine the remaining constants B_n.

$$u(a, \theta) = \sum_{n=1}^{\infty} B_n a^{3n} \sin 3n\theta = f(\theta)$$

Multiply both sides by $\sin 3p\theta$, where p is an integer.

$$\sum_{n=1}^{\infty} B_n a^{3n} \sin 3n\theta \sin 3p\theta = f(\theta) \sin 3p\theta$$

Integrate both sides with respect to θ from 0 to $\pi/3$.

$$\int_0^{\pi/3} \sum_{n=1}^{\infty} B_n a^{3n} \sin 3n\theta \sin 3p\theta \, d\theta = \int_0^{\pi/3} f(\theta) \sin 3p\theta \, d\theta$$

Bring the constants in front of the integral.

$$\sum_{n=1}^{\infty} B_n a^{3n} \int_0^{\pi/3} \sin 3n\theta \sin 3p\theta \, d\theta = \int_0^{\pi/3} f(\theta) \sin 3p\theta \, d\theta$$

Because the sine functions are orthogonal, this integral on the left is zero if $n \neq p$. Only if $n = p$ does the integral yield a nonzero result.

$$B_n a^{3n} \int_0^{\pi/3} \sin^2 3n\theta \, d\theta = \int_0^{\pi/3} f(\theta) \sin 3n\theta \, d\theta$$

$$B_n a^{3n} \left(\frac{\pi}{6}\right) = \int_0^{\pi/3} f(\theta) \sin 3n\theta \, d\theta$$

Therefore,

$$\boxed{B_n = \frac{6}{\pi a^{3n}} \int_0^{\pi/2} f(\theta) \sin 2n\theta \, d\theta.}$$

Part (b)

Substitute the product solution $u(r, \theta) = R(r)\Theta(\theta)$ into the homogeneous boundary conditions.

$$\frac{\partial u}{\partial \theta}(r, 0) = 0 \qquad \rightarrow \qquad R(r)\Theta'(0) = 0 \qquad \rightarrow \qquad \Theta'(0) = 0$$

$$\frac{\partial u}{\partial \theta}\left(r, \frac{\pi}{3}\right) = 0 \qquad \rightarrow \qquad R(r)\Theta'\left(\frac{\pi}{3}\right) = 0 \qquad \rightarrow \qquad \Theta'\left(\frac{\pi}{3}\right) = 0$$

Solve the ODE for Θ.

$$\frac{d^2\Theta}{d\theta^2} = -\lambda\Theta$$

Check to see whether there are positive eigenvalues: $\lambda = \mu^2$.

$$\frac{d^2\Theta}{d\theta^2} = -\mu^2\Theta$$

The general solution can be written in terms of sine and cosine.

$$\Theta(\theta) = C_1 \cos \mu\theta + C_2 \sin \mu\theta$$

Differentiate it with respect to θ.

$$\Theta'(\theta) = \mu(-C_1 \sin \mu\theta + C_2 \cos \mu\theta)$$

Apply the boundary conditions to determine C_1 and C_2.

$$\Theta'(0) = \mu(C_2) = 0$$
$$\Theta'\left(\frac{\pi}{3}\right) = \mu\left(-C_1 \sin \mu\frac{\pi}{3} + C_2 \cos \mu\frac{\pi}{3}\right) = 0$$

This first equation gives $C_2 = 0$, which makes the second equation reduce to $-C_1\mu \sin \mu\frac{\pi}{3} = 0$. To avoid the trivial solution, we insist that $C_1 \neq 0$.

$$\sin \mu\frac{\pi}{3} = 0$$
$$\mu\frac{\pi}{3} = n\pi, \quad n = 1, 2, \ldots$$
$$\mu = 3n$$

There are positive eigenvalues $\lambda = (3n)^2$, and the eigenfunctions associated with them are

$$\Theta(\theta) = C_1 \cos \mu\theta \quad \rightarrow \quad \Theta_n(\theta) = \cos 3n\theta.$$

Using $\lambda = 9n^2$, solve the ODE for R now.

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = 9n^2$$

Expand the left side.

$$\frac{r}{R}(R' + rR'') = 9n^2$$

Multiply both sides by R and bring all terms to the left side.

$$r^2R'' + rR' - 9n^2R = 0$$

This is an equidimensional ODE, so it has solutions of the form $R(r) = r^m$.

$$R = r^m \quad \rightarrow \quad R' = mr^{m-1} \quad \rightarrow \quad R'' = m(m-1)r^{m-2}$$

Substitute these formulas into the ODE and solve the resulting equation for m.

$$r^2 m(m-1)r^{m-2} + rmr^{m-1} - 9n^2 r^m = 0$$

$$m(m-1)r^m + mr^m - 9n^2 r^m = 0$$

$$m(m-1) + m - 9n^2 = 0$$

$$m^2 - 9n^2 = 0$$

$$(m+3n)(m-3n) = 0$$

$$m = \{-3n, 3n\}$$

Two solutions to the ODE are $R = r^{-3n}$ and $R = r^{3n}$. By the principle of superposition, the general solution for R is a linear combination of these two.

$$R(r) = Ar^{-3n} + Br^{3n}$$

Now check to see if zero is an eigenvalue.

$$\frac{d^2\Theta}{d\theta^2} = 0$$

The general solution is a straight line.

$$\Theta(\theta) = C_3 \theta + C_4$$

Apply the two boundary conditions to determine C_3 and C_4.

$$\Theta'(0) = C_3 = 0$$
$$\Theta'\left(\frac{\pi}{3}\right) = C_3 = 0$$

C_4 remains arbitrary.

$$\Theta(\theta) = C_4$$

The trivial solution is not obtained, so zero is an eigenvalue. Using $\lambda = 0$, solve the ODE for R now.

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = 0$$

Multiply both sides by R/r.

$$\frac{d}{dr}\left(r\frac{dR}{dr}\right) = 0$$

Integrate both sides with respect to r.

$$r\frac{dR}{dr} = D_1$$

Divide both sides by r.

$$\frac{dR}{dr} = \frac{D_1}{r}$$

Integrate both sides with respect to r once more.

$$R(r) = D_1 \ln r + D_2$$

Check to see if there are negative eigenvalues: $\lambda = -\gamma^2$.

$$\frac{d^2\Theta}{d\theta^2} = \gamma^2\Theta$$

The general solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$\Theta(\theta) = C_5 \cosh\gamma\theta + C_6 \sinh\gamma\theta$$

Differentiate it with respect to θ.

$$\Theta'(\theta) = \gamma(C_5 \sinh\gamma\theta + C_6 \cosh\gamma\theta)$$

Apply the two boundary conditions to determine C_5 and C_6.

$$\Theta'(0) = \gamma(C_6) = 0$$
$$\Theta'\left(\frac{\pi}{3}\right) = \gamma\left(C_5 \sinh\gamma\frac{\pi}{3} + C_6 \cosh\gamma\frac{\pi}{3}\right) = 0$$

This first equation gives $C_6 = 0$, which makes the second equation reduce to $C_5\gamma\sinh\gamma\frac{\pi}{3} = 0$. No nonzero value of γ can satisfy this equation, so $C_5 = 0$.

$$\Theta(\theta) = 0$$

The trivial solution is obtained, so there are no negative eigenvalues. According to the principle of superposition, the general solution to the PDE is a linear combination of the eigenfunctions over all the eigenvalues.

$$u(r,\theta) = (A_0 \ln r + B_0) + \sum_{n=1}^{\infty}(A_n r^{-3n} + B_n r^{3n}) \cos 3n\theta$$

For the solution to remain finite as $r \to 0$, set $A_0 = 0$ and $A_n = 0$.

$$\boxed{u(r,\theta) = B_0 + \sum_{n=1}^{\infty} B_n r^{3n} \cos 3n\theta}$$

Use the boundary condition at $r = a$ to determine the remaining constants, B_0 and B_n.

$$u(a,\theta) = B_0 + \sum_{n=1}^{\infty} B_n a^{3n} \cos 3n\theta = f(\theta) \tag{1}$$

To find B_0, integrate both sides with respect to θ from 0 to $\pi/3$.

$$\int_0^{\pi/3}\left(B_0 + \sum_{n=1}^{\infty} B_n a^{3n} \cos 3n\theta\right) d\theta = \int_0^{\pi/3} f(\theta)\, d\theta$$

Split up the integral on the left side and bring the constants in front.

$$B_0 \underbrace{\int_0^{\pi/3} d\theta}_{=\,\pi/3} + \sum_{n=1}^{\infty} B_n a^{3n} \underbrace{\int_0^{\pi/3} \cos 3n\theta \, d\theta}_{=\,0} = \int_0^{\pi/3} f(\theta) \, d\theta$$

$$B_0 \left(\frac{\pi}{3}\right) = \int_0^{\pi/3} f(\theta) \, d\theta$$

Therefore,

$$\boxed{B_0 = \frac{3}{\pi} \int_0^{\pi/3} f(\theta) \, d\theta.}$$

To get B_n, multiply both sides of equation (1) by $\cos 3p\theta$, where p is an integer.

$$B_0 \cos 3p\theta + \sum_{n=1}^{\infty} B_n a^{3n} \cos 3n\theta \cos 3p\theta = f(\theta) \cos 3p\theta$$

Integrate both sides with respect to θ from 0 to $\pi/3$.

$$\int_0^{\pi/3} \left(B_0 \cos 3p\theta + \sum_{n=1}^{\infty} B_n a^{3n} \cos 3n\theta \cos 3p\theta \right) d\theta = \int_0^{\pi/3} f(\theta) \cos 3p\theta \, d\theta$$

Split up the integral and bring the constants in front.

$$B_0 \underbrace{\int_0^{\pi/3} \cos 3p\theta \, d\theta}_{=\,0} + \sum_{n=1}^{\infty} B_n a^{3n} \int_0^{\pi/3} \cos 3n\theta \cos 3p\theta \, d\theta = \int_0^{\pi/3} f(\theta) \cos 3p\theta \, d\theta$$

Because the cosine functions are orthogonal, this second integral on the left is zero if $n \neq p$. Only if $n = p$ does the integral yield a nonzero result.

$$B_n a^{3n} \int_0^{\pi/3} \cos^2 3n\theta \, d\theta = \int_0^{\pi/3} f(\theta) \cos 3n\theta \, d\theta$$

$$B_n a^{3n} \left(\frac{\pi}{6}\right) = \int_0^{\pi/3} f(\theta) \cos 3n\theta \, d\theta$$

Therefore,

$$\boxed{B_n = \frac{6}{\pi a^{3n}} \int_0^{\pi/3} f(\theta) \cos 3n\theta \, d\theta.}$$